MidTerm Exam Mathematical Physics, Prof. G. Palasantzas

- Total number of points 100
- 10 points for coming to the exam !
- Justify briefly your answers for all problems

Problem 1 (20 points) Are the infinite series bellow convergent or divergent?

(a: 10 points) $\sum_{n=1}^{\infty} n \sin(1/n)$

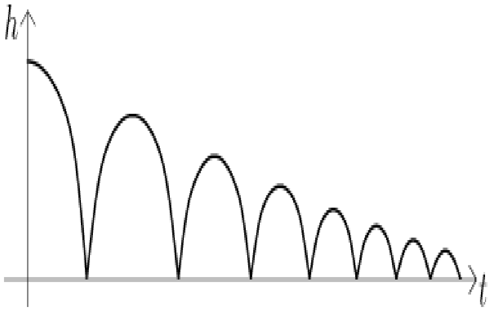
(b: 10 points) $\sum_{n=1}^{\infty} \sin(1/n)$

Problem 2 (20 points)

Find the interval of convergence for the series $\sum_{n=1}^{+\infty} n!(2x-1)^n$

Problem 3 (25 points)

Consider a ball that drops from an initial height h (>0), and that every time it bounces on the ground will lose 40 % of its energy. If the gravitational constant is g (and ignore any friction losses due to the environment), then calculate the total time until the ball stops on the ground (*consider infinite number of bounces until it stops !*).



Problem 4 (25 points)

Suppose a mass m is attached to a spring with spring constant k, and let $k = m\omega^2$. If an external force $F(t) = F_o \cos(\omega t)$ is applied, then we have:

Equation of motion: → $$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

If we assume $c^2 - 4mk < 0$, then show that the motion is described by:

$$x(t) = e^{-(c/2m)t}\left[c_1\cos(\tilde{\omega}t) + c_2\sin(\tilde{\omega}t)\right] + \left(\frac{F_o}{c\omega}\right)\sin(\omega t), \quad \text{with} \quad \tilde{\omega} = \omega\sqrt{1-(c/2m\omega)^2}$$

Problem 1

(a) $\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{1/n}$

$\underset{\text{L' Hopital}}{=\!=} \lim_{n\to\infty} \frac{-\frac{1}{n^2}\cos(1/n)}{-\frac{1}{n^2}} = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1$

Because $\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) \neq 0$ the series diverges

(b) From (a) we have

$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1 > 0$

Becase the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (Harmonic Series) then from the limit comparsion test we have that also the series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ diverges!

Problem 2

If $a_n = n!\,(2x-1)^n$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,(2x-1)^{n+1}}{n!(2x-1)^n}\right| = \lim_{n\to\infty}(n+1)\,|2x-1| \to \infty$ as $n\to\infty$

for all $x \neq \frac{1}{2}$.

(a) Convergent for $x=1/2$

(b) Divergent for $x\neq 1/2$

Problem 3

set $D = 1 - c$ **(c=0.4 in our case)**

Drop from height h: $t_0 = \sqrt{\frac{2h}{g}}$

since we know that $h = \frac{1}{2} g t^2$

First bounce total time $= 2\sqrt{\frac{2hD}{g}}$

Dynamic energy transforms to kinetic and vice versa..

Second bounce total time $= 2\sqrt{\frac{2hD^2}{g}}$

Third bounce total time $: 2\sqrt{\frac{2hD^3}{g}}$

Nth bounce total time $= 2\sqrt{\frac{2hD^n}{g}}$

Total travel time after infinite bounces

$$T = t_0 + \sum_{n=1}^{\infty} 2\sqrt{\frac{2hD^n}{g}} \Rightarrow$$

$$T = \sum_{n=0}^{\infty} \left(2\sqrt{\frac{2h}{g}}\right)(\sqrt{D})^n - \sqrt{\frac{2h}{g}} \Rightarrow$$

$$T = 2\sqrt{\frac{2h}{g}} \underbrace{\sum_{n=0}^{\infty} (\sqrt{D})^n}_{\text{Geometric series } (|\sqrt{D}|<1)} - \sqrt{\frac{2h}{g}} \Rightarrow$$

$$T = 2\sqrt{\frac{2h}{g}} \frac{1}{1-\sqrt{D}} - \sqrt{\frac{2h}{g}} \Rightarrow$$

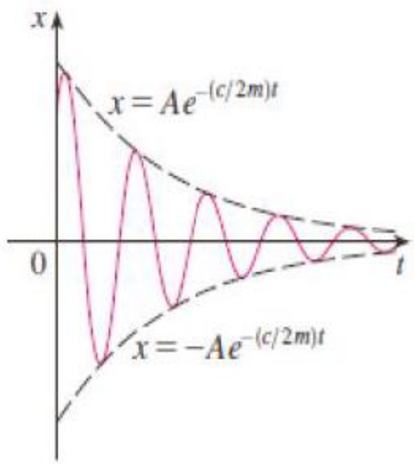
$$T = \sqrt{\frac{2h}{g}} \left(\frac{2}{1-\sqrt{1-c}} - 1 \right) \quad (0<c<1)$$

Problem 4

For the solution of the homogenous

■ CASE III $c^2 - 4mk < 0$ (underdamping)
Here the roots are complex:

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = -\frac{c}{2m} \pm \omega i$$



The solution of the homogenus equation $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$ has the form (assuming $c^2 - 4mk < 0$)

$$x_{homo}(t) = C_1 e^{-t/t_0} \cos \omega' t + C_2 e^{-t/t_0} \sin \omega' t$$

$t_0 = \frac{2m}{c}$ and $\omega' = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$ or

$$\omega' = \omega \sqrt{1 - \left(\frac{c}{2m\omega}\right)^2}$$

In general we have for the particular solution

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_o \cos\omega_o t \quad (1)$$

$$x_p(t) = A(\omega_o)\cos(\omega_o t) + B(\omega_o)\sin(\omega_o t)$$

substitute in (1) $\Rightarrow$

$$m(-A\omega_o^2\cos\omega_o t - B\omega_o^2\sin\omega_o t) + (-AC\omega_o\sin\omega_o t + BC\omega_o\cos\omega_o t) + k(A\cos\omega_o t + B\sin\omega_o t) = F_o\cos\omega_o t \Rightarrow$$

$$[(k-m\omega_o^2)A + CB\omega_o]\cos\omega_o t + [(k-m\omega_o^2)B - CA\omega_o]\sin\omega_o t = F_o\cos\omega_o t \quad (2)$$

$$(k-m\omega_o^2)A + CB\omega_o = F_o \quad (3)$$

$$(k-m\omega_o^2)B - CA\omega_o = 0 \quad (4)$$

since $\omega = \omega_o$ (and as a result $k - m\omega^2 = 0$) we obtain after substition into the equation of motion : $c\omega B = F_o$ and $A = 0$

$$\Rightarrow x_p(t) = \left(\frac{F_o}{c\omega}\right)\sin(\omega t)$$

Full solution:

$$x(t) = e^{-(c/2m)t}[c_1\cos(\tilde{\omega}t) + c_2\sin(\tilde{\omega}t)] + \left(\frac{F_o}{c\omega}\right)\sin(\omega t)$$

$$\tilde{\omega} = \omega\sqrt{1-(c/2m\omega)^2}$$